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SOME BASIC IDEAS OF ROUGH ANALYSIS

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ABSTRACT. The classical analysis is often based on fine behaviors which are valid *for all* points of some subsets, even if some distance tends to zero. Since many things of the material universe and many objects represented by digital computers cannot satisfy such *for-all*-requirements, the so-called rough analysis is developed as an approach to such rough worlds. In this context, certain properties are only required for distances greater than some given roughness degree. For instance, a sequence (x_i) in a normed space is said to be roughly convergent to x_* w.r.t. the roughness degree $r \geq 0$ if for all $r_\varepsilon > r$ there exists an $i_\varepsilon \in \mathbb{N}$ such that $i \geq i_\varepsilon$ implies $\|x_i - x_*\| < r_\varepsilon$. A function $f : D \to \mathbb{R}$ is called roughly convex w.r.t. the roughness degree r if some convexity condition is fulfilled between any pair of points $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| \geq r$. To illustrate the main ideas and some advantages of this approach, in this paper we summarize some results concerned with rough convergence and rough continuity, fixed-point theorems of roughly continuous mappings, rough convexity and its application to global optimization.

1. Introduction

Several fundamental notions of analysis are defined in connection with some requirements "*for all...*". For instance, a sequence (x_i) in some normed space is said to be *convergent* to x_* denoted by $x_i \to x_*$, if

$$\forall \varepsilon > 0 \quad \exists\, i_\varepsilon \in \mathbb{N} : \quad i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < \varepsilon.$$

Based on the convergence, further important notions are introduced, such as continuity, derivative, integral, et cetera. When can we have such a convergence, where the corresponding distance tends to zero? It is possible as long as the considered points are ideal mathematical objects with radius zero, as Euclid assumed in the first book of his historical "Elements":

"*A point is that which has no parts. A line is length without width.*"

But for "real points" which describe real objects (such as tiny particles or even huge planets) by identifying them with their center points, the distance cannot be less than the sum of their radii, which is certainly greater than zero. Thus, the above ideal convergence is not always suitable for those "real points".

Let us consider a convergent (or continuous) process from practical point of view. In general, it often cannot be measured, or calculated, or modeled, or simulated exactly, especially if it is done by digital computers which are only able to represent finitely many rational numbers. Therefore, we obtain an approximation for which the original convergence (or continuity) fails to maintain. It is quite natural to ask in which sense such an approximation is convergent (or continuous, resp.)?

Behind the above requirement "$\forall \varepsilon > 0$", there is a crucial quantity called infinitesimal, which is a significant achievement of mathematics. But one has to pay attention that this infinitesimal is not always applied exactly in mathematical sense, as declared by L. D. Landau and E. M. Lifschitz in their famous Course of Theoretical Physics (Vol. 6: Fluid Mechanics [26]):

> *"... any small volume element (dV) in the fluid is always supposed so large that it still contains a very great number of molecules. Accordingly, when we speak of infinitely small elements of volume, we shall always mean those which are `physically' infinitely small, i.e. very small compared with the volume of the body under consideration, but large compared with the distances between the molecules."*

Hence, such an infinitesimal in fluid dynamics is not exactly a mathematical infinitesimal, which must be less than every positive real number.

Another example of *for-all*-requirements is that a subset S of some linear space is named convex if

$$\forall x_0, x_1 \in S \ \forall \lambda \in [0,1] : (1-\lambda) x_0 + \lambda x_1 \in S.$$

Indisputably, convexity is a substantial notion in functional analysis, geometry and optimization. But it is hard to find a material object which is really convex because most of physical things consist of separate tiny particles. From this point of view, such objects cannot be connected either. Therefore, if we see something convex or connected in our universe then it is almost certainly an inexact image. Even if we try to represent a convex set by a digital computer, we often obtain only a finite set of discrete points.

We see that the real world and the computer world are not fine enough to satisfy such *for-all*-requirements. Maybe, *rough worlds* also need a corresponding *rough analysis*? Motivated by this idea, we like to develop a theory which should build a bridge between rough worlds and the classical analysis (say, the *fine analysis*). To meet as many objects of rough worlds as possible, if necessary, we direct at modest targets which are possibly easier to attain, instead of following ideal ones, whose existence is difficult to prove, and which are hardly reachable (exactly). Especially, we like to create suitable prisms for looking complicated objects in some simpler manner. It is worth mentioning that some new results can be obtained, which are not present in the classical analysis.

2. Rough convergence

Throughout this paper, $(X, ||\cdot||)$ is a normed linear space and r and ρ are two given non-negative real numbers.

A sequence $(x_i) \subset X$ is said to be r-*convergent* to some point $x_* \in X,$ denoted by $x_i \xrightarrow{r} x_*,$ if

$$\forall \varepsilon > 0 \quad \exists\, i_\varepsilon \in \mathbb{N} : i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < r + \varepsilon,$$

i.e.,

$$\limsup_{i \to \infty} \|x_i - x_*\| \leq \varepsilon.$$

Since such an r-*limit point* x_* is no more unique, we have to study the r-*limit* set in some given subset $S \subset X$ defined by

$$\mathrm{LIM}^{S,r}\, x_i := \{x_* \in S : x_i \xrightarrow{r} x_*\}.$$

Interesting particular cases are for instance $S = X$ and $S = (x_i)$ (i.e., $S = \{x_i : i \in \mathbb{N}\}$).

For $S = X$ we write $\mathrm{LIM}^r\, x_i := \mathrm{LIM}^{X,r}\, x_i.$ If then $\mathrm{LIM}^r\, x_i \neq \emptyset$ then (x_i) is said to be r-*convergent* and r is called a *convergence degree* of (x_i).

Similarly, $(x_i) \subset X$ is ρ-*Cauchy* and ρ is a *Cauchy degree* of (x_i) if

$$\forall \varepsilon > 0 \quad \exists\, i_\varepsilon : i, j \geq i_\varepsilon \Rightarrow \|x_i - x_j\| < \rho + \varepsilon.$$

For $r = 0$ and $\rho = 0$ we have the definition of classical convergence and of Cauchy sequence again.

To justify the introduction of rough convergence and rough Cauchy sequence, let us consider an example from numerical point of view. As mentioned in Section 1, an originally convergent sequence (y_i) (with $y_i \to x_*$) often cannot be determined exactly, but only approximated by some sequence (x_i) satisfying $\|x_i - y_i\| \leq \Delta$ for all $i,$ where $\Delta > 0$ is an upper bound of approximation errors. In general, (x_i) is no more convergent, but

$$\|x_i - x_*\| \leq \|x_i - y_i\| + \|y_i - x_*\| \leq \Delta + \|y_i - x_*\|$$

yields that it is r-convergent for $r = \Delta.$ Similarly, if an originally Cauchy sequence (y_i) is approximated by a sequence (x_i) with $\Delta > 0$ as an upper bound of approximation errors, then, for all $\varepsilon > 0$ there exists i_ε such that

$$i, j \geq i_\varepsilon \quad \Rightarrow \quad \|x_i - x_j\| \leq \|x_i - y_i\| + \|y_i - y_j\| + \|y_i - x_j\| < 2\Delta + \varepsilon,$$

i.e., (x_i) is a ρ-Cauchy sequence for $\rho = 2\Delta.$

In [34]-[37] we investigated some properties of r-limit sets, the relation between this rough convergence and other convergence notions, the dependence of $\mathrm{LIM}^r\, x_i$ on $r,$ and the dependence of convergence degree r on Cauchy degree $\rho.$ Next, let us summarize some of these results.

2.1. Properties of r-limit sets

As basic properties of classical convergence, we have:

- The limit point of a convergent sequence is unique;
- Each subsequence of a convergent sequence converges to the same limit point;
- A convergent sequence is bounded;
- A sequence in a relatively compact set possesses convergent subsequences.

The analogy of these properties for rough convergence is given in the following.

Proposition 2.1. ([34], [37])

(a) *The diameter of an r-limit set is not greater than $2r$. In general, there is no smaller bound.*

(b) *If (x_i') is a subsequence of (x_i), then $\mathrm{LIM}^r\, x_i \subseteq \mathrm{LIM}^r\, x_i'$.*

(c) *A sequence (x_i) is bounded if and only if there exists an $r \geq 0$ such that $\mathrm{LIM}^r\, x_i \neq \emptyset$.*

(d) *If $C \subset X$ is relatively compact and $r \geq 0$, then each sequence (x_i) in*

$$C + \overline{B}_r(0) = \{x + z : x \in C, z \in \overline{B}_r(0)\}$$

(where $\overline{B}_r(y) := \{z \in X : \|z - y\| \leq r\}$) contains a subsequence (x_{i_j}) satisfying

$$\mathrm{LIM}^r\, x_{i_j} \neq \emptyset \text{ and } \mathrm{LIM}^{(x_{i_j}),r'}\, x_{i_j} \neq \emptyset \text{ for every } r' > r.$$

In particular, this holds true for every $r > 0$ and every bounded sequence in a finite dimensional normed space.

Since the limit point of a convergent sequence is unique, there is nothing to say about geometrical and topological properties of the set of limit points. But, for $r > 0$, an r-limit set may contain several points. Therefore, it is reasonable to investigate such properties of them.

Proposition 2.2. ([34], [37]) *$\mathrm{LIM}^r\, x_i$ is closed and convex. If the normed space X is uniformly convex, then $\mathrm{LIM}^r\, x_i$ is strictly convex, i.e., $y_0, y_1 \in \mathrm{LIM}^r\, x_i$ and $y_0 \neq y_1$ imply*

$$y_\lambda := (1-\lambda)y_0 + \lambda x_1 \in \mathrm{int}(\mathrm{LIM}^r\, x_i) \text{ for all } \lambda \in (0,1).$$

2.2. Relations to other convergence notions

As mentioned above, if (x_i) is an approximation of a convergent sequence $y_i \to x_*$ with r as an upper bound of approximation error then (x_i) is r-convergent to x_*. Conversely, if (x_i) is r-convergent to x_* then there exists a sequence (y_i) near (x_i) (i.e., $\|x_i - y_i\| \leq r$ for all i) which converges (in the classical sense) to x_*. This equivalent relation is contained in the following more general result.

Proposition 2.3. ([34]) *Suppose* $r_1 \geq 0$ *and* $r_2 > 0$. *A sequence* (x_i) *in* X *is* $(r_1 + r_2)$-*convergent to* x_* *if and only if there exists a sequence* (y_i) *in* X *such that* $y_i \xrightarrow{r_1} x_*$ *and* $\|x_i - y_i\| \leq r_2,\ i = 1,2,\ldots$

The next two propositions state further close relations between the classical convergence and the rough convergence.

Proposition 2.4. ([37]) *Let* $r > 0$ *and* $(x_i) \subset X$.

(a) *If* $(x_i) \subset X$ *converges to* x_* *then* $\mathrm{LIM}^r x_i = \overline{B}_r(x_*) = \{z \in X : \|z - x_*\| \leq r\}$.

(b) *If* (x_i) *is contained in some compact set of* X *and if* $\mathrm{LIM}^r x_i = \overline{B}_r(x_*)$ *then* (x_i) *converges to* x_*.

(c) *If* X *is uniformly convex and if there are* $y_0, y_1 \in \mathrm{LIM}^r x_i$ *satisfying* $\|y_0 - y_1\| = 2r$ *then* (x_i) *converges to* $2^{-1}(y_0 + y_1)$.

Proposition 2.5. ([37]) *Let* C *be the set of cluster points of* (x_i).

(a) *If* $C \neq 0$ *then* $\mathrm{LIM}^r x_i \subseteq \bigcap_{c \in C} \overline{B}_r(c)$.

(b) *If* (x_i) *is contained in some compact set of* X *then*

$$\mathrm{LIM}^r x_i = \bigcap_{c \in C} \overline{B}_r(c) = \{x_* \in X : C \subseteq \overline{B}_r(x_*)\}.$$

Recall that if $(K_i)_{i \in \mathbb{N}}$ is a sequence of subsets of a metric space X then the subsets

$$\operatorname*{Lim\,sup}_{i\to\infty} K_i := \{x \in X \mid \liminf_{i\to\infty} d(x, K_i) = 0\},$$
$$\operatorname*{Liminf}_{i\to\infty} K_i := \{x \in X \mid \lim_{i\to\infty} d(x, K_i) = 0\}$$

are called *upper* or *lower limit* of this sequence, respectively (see [2]). Using these notions, the relation of rough convergence to set convergence in set-valued analysis can be described as follows.

Proposition 2.6. ([37])

(a) $\mathrm{LIM}^r x_i = \mathrm{Liminf}_{i\to\infty} \overline{B}_r(x_i)$.

(b) *If* $\mathrm{Limsup}_{i\to\infty}\{x_i\} \neq \emptyset$ *then* $\mathrm{LIM}^r x_i \subseteq \bigcap_{c \in \mathrm{Limsup}_{i\to\infty}\{x_i\}} \overline{B}_r(c)$, *where the equality holds if* (x_i) *is contained in some compact set of* X.

2.3. The dependence of $\mathrm{LIM}^r x_i$ on r

Let (x_i) be an arbitrary sequence in X. It follows immediately from definition

$$\mathrm{LIM}^{r_1} x_i \subseteq \mathrm{LIM}^{r_2} x_i \text{ if } 0 \leq r_1 < r_2.$$

This monotonicity is included in the following proposition.

Proposition 2.7. ([37])

(a) *If* $r \geq 0$ *and* $\sigma > 0$ *then* $\mathrm{LIM}^r\, x_i \subseteq \mathrm{LIM}^r\, x_i + \overline{B}_\sigma(0) \subseteq \mathrm{LIM}^{r+\sigma}\, x_i$.

(b) *If* X *is uniformly convex and if* y *is an interior point of* $\mathrm{LIM}^r\, x_i$ *then there exists an* $r' \in [0, r)$ *such that* $y \in \mathrm{LIM}^{r'}\, x_i$.

(c) *If* (x_i) *is contained in some compact set of* X *then* $\overline{B}_\sigma(y) \subseteq \mathrm{LIM}^r\, x_i$ implies $y \in \mathrm{LIM}^{r-\sigma}\, x_i$.

For

$$\overline{r} := \inf\{r \in \mathbb{R}_+ : \mathrm{LIM}^r\, x_i \neq \emptyset\}$$

the above result yields

$$\mathrm{LIM}^r\, x_i \begin{cases} = \emptyset \text{ for } r < \overline{r} \\ \neq \emptyset \text{ for } r > \overline{r} \end{cases} \quad \text{and} \quad \mathrm{int}(\mathrm{LIM}^r\, x_i) \neq \emptyset \quad \text{for} \quad r > \overline{r}.$$

Moreover, we have

Proposition 2.8. ([34])

$$\mathrm{cl}\Big(\bigcup_{0 \leq r' < r} \mathrm{LIM}^{r'}\, x_i\Big) \subseteq \mathrm{LIM}^r\, x_i = \bigcap_{r' > r} \mathrm{LIM}^{r'}\, x_i.$$

If $r \neq \overline{r}$ *then* $\mathrm{cl}(\bigcup_{0 \leq r' < r} \mathrm{LIM}^{r'}\, x_i) = \mathrm{LIM}^r\, x_i$.

Especially, we are interested in what happens at $r = \overline{r}$? The answer is not obvious and given for some particular cases.

Proposition 2.9. ([34], [37])

(a) *If* X *is a reflexive* B*-space then* $\mathrm{LIM}^{\overline{r}}\, x_i \neq \emptyset$.

(b) *If* (x_i) *is contained in some compact set then* $\mathrm{int}(\mathrm{LIM}^{\overline{r}}\, x_i) \neq \emptyset$.

(c) *If* X *is a uniformly convex* B*-space then* $r = \overline{r}$ *if and only if* $\mathrm{LIM}^r\, x_i$ *is a singleton.*

(d) *If* X *is finite dimensional then the set-valued mapping* $r \mapsto \mathrm{LIM}^r\, x_i$ *is continuous on* $(\overline{r}, +\infty)$, *and*

$$r = \overline{r} \text{ if and only if } \mathrm{LIM}^r\, x_i \neq \emptyset \text{ and } \mathrm{int}(\mathrm{LIM}^r\, x_i) = \emptyset.$$

2.4. The dependence of convergence degree on Cauchy degree

What is the (generally) smallest convergence degree of ρ-Cauchy sequences in some normed space X ? An important tool for answering this question is the *Jung constant* of X defined by

$$J(X) := \sup\Big\{\tfrac{2 r_X(S)}{d(S)} : S \subset X, 0 < d(S) < \infty\Big\},$$

where

$$d(S) := \sup_{x,y\in S} \|x-y\|, \;\; r_X(S) := \inf_{x\in X} \sup_{y\in S} \|x-y\|$$

denote the *diameter* and the *radius*, respectively, of S in X. Obviously, $1 \le J(X) \le 2$.

This constant was investigated firstly by Jung [20], who showed $J(\ell_2^n) = \left(\frac{2n}{n+1}\right)^{1/2}$ for the n-dimensional Euclidean space ℓ_2^n.

Due to Bohnenblust [6], $J(X) \le 2n/(n+1)$ if X is an n-dimensional Minkowski space. Later Leichtweiss [27] and Grünbaum [12] characterized those Minkowski spaces for which $J(X) = 2n/(n+1)$.

Ball [4] and Pichugov [51] proved

$$J(\ell_p) = J(L_p) = \max\{2^{1/p}, 2^{1-1/p}\}, \;\; 1 \le p < \infty.$$

Davis [11] showed that $J(X) = 1$ if and only if X is a $\mathcal{P}_1$-space. In particular, $J(l_\infty) = J(L_\infty) = 1$.

Due to Amir [1], if a compact Hausdorff space T is not extremally disconnected, then for every finite-codimensional subspace X of $C(T)$ we have $J(X) = 2$. Moreover, it was shown by Maluta [28] and by Amir [1] that $J(X) = 2$ if X is a nonreflexive Banach space.

Theorem 2.10. ([37]) *Let $(x_i) \subset X$ be ρ-Cauchy for some $\rho \ge 0$ and $J(X)$ be the Jung constant of X. Then (x_i) is r-convergent for all $r > 2^{-1}J(X)\rho$. If $\dim X < \infty$ then (x_i) is r-convergent for all $r \ge 2^{-1}J(X)\rho$.*

Using concrete values of the Jung constant, we can find convergence degrees of ρ-Cauchy sequences in particular normed spaces. For instance, each ρ-Cauchy sequence (x_i) in ℓ_2^n is r-convergent for all $r \ge (\frac{n}{2(n+1)})^{1/2}\rho$. Moreover, for this special space, we obtained a stronger result.

Proposition 2.11. ([34]) *Let M be a convex (not necessarily closed) subset of the n-dimensional Euclidean space, and (x_i) be a ρ-Cauchy sequence in M (for instance $M = \operatorname{conv}\{x_i : i \in \mathbb{N}\}$). Then*

$$\operatorname{LIM}^{M,r} x_i \ne \emptyset \quad \text{for} \quad r \ge \left(\frac{n}{2(n+1)}\right)^{1/2} \rho.$$

3. Rough continuity

As usual, a mapping $f : X \to Y$ is said to be *continuous* provided $x' \to x$ (in X) always implies $f(x') \to f(x)$ (in Y), where the two latter arrows denote the classical convergence in the corresponding spaces. Replacing the classical convergence by the rough convergence, we obtain the so-called *rough continuity*, or r_X-r_Y-*continuity*, where r_X and r_Y denote the convergence degrees in X and Y, respectively.

For $r_X = 0$, f is called 0-r_Y-*continuous* if $x' \to x$ always implies $f(x') \xrightarrow{r_Y} f(x)$.

By definition, $x' \xrightarrow{r_X} x$ means $x' \to \overline{B}_{r_X}(x)$, i.e., the classical convergence of x' to the ball $\overline{B}_{r_X}(x)$. Hence, for $r_X \geq r_Y = 0$, f is called r_X-0-*continuous* if $x' \xrightarrow{r_X} x$ implies $f(x') \to f(\overline{B}_{r_X}(x))$.

In general, f is called r_X-r_Y-*continuity* if $x' \xrightarrow{r_X} x$ implies $f(x') \xrightarrow{r_Y} f(\overline{B}_{r_X}(x))$, i.e.

$$\limsup_{x' \xrightarrow{r_X} x} \left(\operatorname{dist}(f(x'), f(\overline{B}_{r_X}(x)))\right) \leq r_Y,$$

where $\operatorname{dist}(.,.)$ denotes the infimal distance between a point and a set.

For $r_X = r_Y = 0$ the r_X-r_Y-continuity is just the classical continuity.

To state an example of 0-r_Y-continuous mappings, let us consider the parametric optimization problem

$$\text{minimize } f(t,u) \text{ subject to } u \in U,$$

where $f : T \times U \to \mathbb{R}$, and T and U are subsets of suitable normed spaces. The following is contained in a more general result in [45].

Proposition 3.1. *Suppose f is continuous and $f(t,\cdot)$ is strictly r-convexlike, i.e., if $\|u_0 - u_1\| > r$ then there exists a $\lambda \in (0,1)$ such that*

$$f(t,(1-\lambda)u_0 + \lambda u_1) < (1-\lambda)f(t,u_0) + \lambda f(t,u_1).$$

Suppose U is compact and $u^* : T \to U$ satisfies $f(t,u^*(t)) = \min_{u \in U} f(t,u)$ for all $t \in T$. Then u^* is 0-r-continuous.

As an example of use, we studied the transportation problem

$$\int_{t_0}^{t_f} \left(L_1(t,x(t)) + L_2(t,u(t))\right) dt \to \min!$$

$$\dot{x}(t) = d(t) - u(t),\ 0 \leq u(t) \leq \beta,$$

$$0 \leq x(t) \leq \alpha,\ x(t_0) = x_0,\ x(t_f) = x_f,$$

where the state function x and the control function u describe the stock and the transport amount, respectively. Suppose that each vehicle can carry one good unit and the transport cost $L_2(t,\cdot)$ is strictly r-convexlike for $r = 1$. Under some additional assumptions, it was shown in [45] that the optimal transport amount u^* is 0-1-continuous, i.e., the corresponding optimal number of used vehicles changes at most by 1 at each point of time.

Note that strictly and roughly convexlike functions were investigated in [41].

Examples of r-0- continuous mappings can be found in the class of linear operators $f : X \to Y$. It is well know that such an operator is not necessarily continuous in the classical sense if $\dim X = \infty$, even for $\dim Y = 1$. Therefore, the next result is a particular one of rough analysis.

Theorem 3.2. ([35]) *Let X and Y be two normed spaces. If $\dim X < \infty$ and $r > 0$ then every linear operator $f : X \to Y$ is r-0- continuous, i.e.,*

$$\operatorname{dist}(x, \overline{B}_r(0)) \to 0 \textit{ implies } \operatorname{dist}(f(x), f(\overline{B}_r(0))) \to 0$$

(even if $f(\overline{B}_r(0))$ is unbounded).

Let us mention an important property of continuous functions: A continuous function $f : D \to \mathbb{R}$ on a compact set D attains its maximum and minimum. To extend the applicability, this Weierstrass theorem is often formulated as follows: A lower semi-continuous function $f : D \to \mathbb{R}$ defined by

$$\liminf_{x' \to x} f(x') \geq f(x), \ \forall x \in D \tag{3.1}$$

on a compact set D attains its minimum, i.e.,

$$\exists x_* \in D : f(x_*) = \inf_{x \in D} f(x).$$

Note that a convex function $f : D \to \mathbb{R}$ (on a convex set D) is not necessarily lower semi-continuous, even if D is a subset of a finite dimensional space. For instance, if D is the unit ball of a Euclidean space, z is a boundary point of D, and

$$f(x) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{if } x \in D \setminus \{z\}, \end{cases}$$

then f is convex on D, but it is not lower semicontinuous at z. This fact changes in rough context.

$f : D \subset X \to \mathbb{R}$ is called r- *lower semicontinuous* if

$$\liminf_{x' \xrightarrow{r} x} f(x') \geq \inf_{z \in \overline{B}_r(x) \cap D} f(z), \ \forall x \in D. \tag{3.2}$$

Obviously, if $r = 0$ then (3.1) and (3.2) are the same. But the more interesting case is $r > 0$.

Theorem 3.3. *Let $D \subset X$ and $f : D \to \mathbb{R}$ be convex. Then, for an arbitrary $r > 0$, f is r- lower semicontinuous.*

The last result is also an example for the advantages of the rough way of observation. Using the rough lower semicontinuity, we can generalize the Weierstrass theorem as follows.

Theorem 3.4. *Suppose that* $D \subset X$ *is compact and* $r \geq 0$, *or* $D \subset C + \overline{B}_r(0)$ *for some relatively compact* $C \subset X$ *and* $r > 0$. *Let* $f : D \subset X \to \mathbb{R}$ *be* r-*lower semicontinuous. Then*

$$\exists x_* \in D : \inf_{x \in \overline{B}_r(x*) \cap D} f(x) = \inf_{x \in D} f(x).$$

Two further kinds of rough continuity and their application will be explained in the coming section.

4. Roughly fixed-point theorems

The existence of solutions of equations is a central problem of mathematics. A main tool for this problem is given by fixed-point theorems, which deal with the existence of fixed (or invariant) points of a mapping $T : M \to M$ defined by

$$x_* = Tx_*, \; x_* \in M.$$

Two fundamental fixed-point principles are the following (see [59]).

Theorem 4.1. (Banach Fixed-Point Theorem [5]) *Let* M *be a closed nonempty set in a complete metric space* (X, d), *and* $T : M \to M$ *be* k-*contractive, i.e.,*

$$d(Tx, Ty) \leq kd(x, y), \; \forall x, y \in M$$

for a fixed $k \in [0,1)$. *Then* T *has exactly one fixed point on* M, *and for an arbitrary choice of initial point* $x_0 \in M$, *the sequence* (x_i) *of successive approximations*

$$x_{i+1} = Tx_i, \; i = 0,1,2,\ldots \tag{4.1}$$

converges to the fixed point x_*.

Theorem 4.2. (Brower Fixed-Point Theorem [7]) *Suppose that* M *is a nonempty convex compact set of* $\mathbb{R}^n$, $n \geq 1$, *and that* $T : M \to M$ *is continuous. Then* T *has a fixed point.*

There are numerous generalizations of the above theorems. For instance, the Brower fixed-point theorem was generalized to normed linear spaces by Schauder [55], and to locally convex topological vector spaces by Tychonov [58]. A common assumption of those theorems is the continuity of mapping T. What happens if T is not continuous? In general, we cannot expect such a mapping T to admit fixed points but only γ-*fixed* or γ-*invariant points* defined by

$$d(x_*, Tx_*) \leq \gamma.$$

It is of interest to determine the minimal (or a possibly small) $\gamma \geq 0$ such that a given T has γ-fixed points. This problem was investigated in [8], [9], [10], [21], [23], [38], [40], [50],....

An important tool for our investigation is the *self-Jung constant* of a normed linear space X defined by

$$J_s(X) := \sup\left\{\frac{2r_{\operatorname{conv} S}(S)}{d(S)} : S \subset X,\ \ 0 < d(S) < \infty\right\},$$

where $d(S) = \sup_{x,y \in S} \|x - y\|$ is the diameter and

$$r_{\operatorname{conv} S}(S) := \inf_{x \in \operatorname{conv} S} \sup_{y \in S} \|x - y\|$$

denotes the *self-radius* of S. Obviously, $1 \le J_s(X) \le 2$ holds true for any normed space X.

Due to Klee [22], $r_{\operatorname{conv} S}(S) = r_X(S)$ for all bounded $S \subset X$ is equivalent to X having dimension ≤ 2 or being an inner-product space. Therefore, it follows from the results of Jung [20] and Routledge [54] that

$$J_s(\ell_2^n) = \left(\frac{2n}{n+1}\right)^{1/2} \text{ and } J_s(\ell_2) = \sqrt{2}. \tag{4.2}$$

Amir [1] proved

$$J_s(X) \le \frac{2n}{n+1} \text{ if } \dim X = n. \tag{4.3}$$

Since $r_X(S) \le r_{\operatorname{conv} S}(S)$, due to Bohnenblust [6], Grünbaum [12], and Leichtweiss [27], there is no smaller upper bound which holds true for all n- dimensional normed spaces.

Pichugov [51] showed

$$J_s(\ell_p) = J_s(L_p[0,1]) = \max\{2^{1/p}, 2^{1-1/p}\}, 1 \le p < \infty.$$

Self-Jung constant for some extreme cases was given by Maluta [28], for instance

$$J_s(X) = 1 \text{ for } X = (\mathbb{R}^2, ||\cdot||_\infty)$$

and

$$J_s(X) = 2 \text{ if } X \text{ is a nonreflexive Banach space.}$$

Let us now consider r- *roughly* k- *contractive* mappings $T : M \to M$ defined by

$$|| T_x - T_y || \le k || x - y || + r, \ \forall x, y \in M$$

for a fixed $k \in [0,1)$ and a fixed $r \ge 0$. Such a mapping may arise quite naturally. For instance, an original k- contractive mapping $\hat{T}$ often cannot be determined exactly, but only approximated by a mapping $\hat{T}$ satisfying

$$\sup_{x \in M} || Tx - \hat{T}x || \le \frac{r}{2},$$

i.e., $r/2$ is an upper bound of approximation error. Then

$$\left\|\hat{T}x-\hat{T}y\right\| \le \left\|\hat{T}x-Tx\right\| + \|Tx-Ty\| + \left\|Ty-\hat{T}y\right\| \le k\|x-y\| + r,$$

i.e., $\hat{T}$ is r- roughly k- contractive.

By applying successive approximations to r- roughly k- contractive mappings, we obtain the following result, which was actually stated for a metric space (M,d) in [50].

Theorem 4.3. ([50]) *Let $T: M \to M$ be an r- roughly k- contractive mapping, where $r \ge 0$ and $k \in (0,1)$ are given. Suppose $x_0 \in M$ and*

$$a := \|x_0 - Tx_0\| - \frac{r}{1-k} > 0.$$

(a) *If $\gamma > r/(1-k)$ and $i \ge \log_k\left(\left(\gamma - \frac{r}{1-k}\right)a^{-1}\right)$ then x_i determined by* (4.1) *is a γ-fixed point of T.*

(b) *If $x_* \in M$ is a cluster point of the sequence (x_i) then it is a γ-fixed point of T with $\gamma = r/(1-k)$.*

(c) *For every $\gamma > 0$, the set I_γ of all γ-fixed points of T is bounded. If $\gamma \ge r/(1-k)$ then I_γ is invariant under T, i.e., $TI_\gamma \subset I_\gamma$.*

Obviously, if k is very close to 1 then $r/(1-k)$ is very large. But this result cannot be improved anymore, as showed in [50] by considering the r- roughly k- contractive mapping $T: M_1 \cup M_2 \to M_1 \cup M_2$ defined by

$$Tx = \begin{cases} \frac{r}{2} - kx & \text{if } x \in M_1 = (-\infty, -r/2(1-k)] \\ -\frac{r}{2} - kx & \text{if } x \in M_2 = [r/2(1-k), \infty), \end{cases}$$

which admits no γ- fixed points with $\gamma < r/(1-k)$ because $|x - Tx| \ge r/(1-k)$ for all $x \in M_1 \cup M_2$.

Even if γ- fixed points with $\gamma < r/(1-k)$ do exist, the iteration (4.1) is not suitable to approximate them, as pointed out in [50] by considering $T: \mathbb{R} \to \mathbb{R}$ satisfying

$$Tx = \begin{cases} \frac{r}{2} - kx & \text{if } x \le 0 \\ -\frac{r}{2} - kx & \text{if } x > 0. \end{cases}$$

This mapping is r- roughly k- contractive, and for any $\gamma \ge \frac{1}{2}r$, each x satisfying

$$-\frac{\gamma - \frac{r}{2}}{1+k} \le x \le \frac{\gamma - \frac{r}{2}}{1+k}$$

is a γ- fixed point of T. But, for any starting point $x_0 \in \mathbb{R}$, the sequence (x_i) determined by (4.1) has only two cluster points $\pm \frac{r}{2(1-k)}$ and

$$T\frac{r}{2(1-k)} = \frac{-r}{2(1-k)},\ T\frac{-r}{2(1-k)} = \frac{r}{2(1-k)}$$

implies

$$\left|\frac{r}{2(1-k)} - T\frac{-r}{2(1-k)}\right| = \left|\frac{-r}{2(1-k)} - T\frac{-r}{2(1-k)}\right| = \frac{r}{1-k}.$$

In particular, if $-\frac{r}{2(1-k)} < x_0 < \frac{r}{2(1-k)}$ then

$$|x_i - Tx_i| < |x_{i+1} - Tx_{i+1}|,\ i = 0,1,2,...,$$

i.e., the more we try to approximate by successive approximation, the worse becomes the result. From the computational point of view, this remark is rather important, since one has often to do with approximated r- roughly k- contractive mappings instead of original k- contractive ones.

A smaller invariant degree γ is only guaranteed if the domain M is assumed to be convex. By using the self-Jung constant of X, we get

Theorem 4.4. ([40]) *Let $T : M \to M$ be an r- roughly k- contractive mapping on a closed and convex subset M of some n- dimensional normed space X. Then*

$$\forall \varepsilon > 0 \quad \exists x_* \in M : \|x_* - Tx_*\| < \frac{1}{2} J_s(X) r + \varepsilon.$$

If $\dim X = 1$, or X is some two-dimensional strictly convex normed space, or X is a Euclidean space then

$$\exists x_* \in M : \|x_* - Tx_*\| \le \frac{1}{2} J_s(X) r.$$

For $\dim X = n$, the above theorem and (4.3) yield

$$\forall \varepsilon > 0 \quad \exists x_* \in M : \|x_* - Tx_*\| < \frac{n}{n+1} r + \varepsilon.$$

In particular, if X is the n- dimensional Euclidean space ℓ_2^n, then it follows from (4.2) that

$$\exists x_* \in M : \|x_* - Tx_*\| \le \left(\frac{n}{2(n+1)}\right)^{\frac{1}{2}} r.$$

This result is the same as given in [38] and [50].

To generalize the Brower fixed-point theorem, let us define two kinds of roughly continuous mappings.

For a given $r \geq 0,$ a mapping $T: M \to M$ on some subset M of a normed linear space X is said to be *around* r-*continuous* if for all $x \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\|Ty - Tz\| < r + \varepsilon$ holds whenever $y, z \in M,$ $\|y - x\| < \delta,$ and $\|z - x\| < \delta.$ If δ does not depend on x then T is called *uniformly* r-*continuous*. Note that an r-roughly k-contractive mapping is uniformly r-continuous. In the following roughly fixed-point theorem, if $T: M \to M$ is uniformly r-continuous then M is not required to be closed.

Theorem 4.5. ([36]) *Let M be a nonempty convex subset of a B-space X and $T: M \to M$. For a given $r \geq 0,$ suppose that one of the following is true:*

(i) *M is compact and T is around r-continuous;*

(ii) *M is relatively compact and T is uniformly r-continuous;*

(iii) *M is closed, $T(M)$ is relatively compact, and T is around r-continuous;*

(iv) *$T(M)$ is relatively compact and T is uniformly r-continuous.*

Then we have:

(a) *For all $\rho > 0,$ there exists $x_* \in M$ such that*

$$\|x_* - Tx_*\| < \frac{1}{2} J_s(X) r + \rho.$$

(b) *If $\dim X < \infty$ and*

$$J_s(X') < J_s(X) \text{ for each proper subspace } X' \subset X, \tag{4.4}$$

then there exists $x_* \in M$ satisfying

$$\|x_* - Tx_*\| \leq \frac{1}{2} J_s(X) r.$$

(c) *If $\dim X = \infty$ and*

$$J_s(X') < J_s(X) \text{ for each finite dimensional subspace } X' \subset X, \tag{4.5}$$

then there exists $x_* \in M$ satisfying

$$\|x_* - Tx_*\| < \frac{1}{2} J_s(X) r.$$

(d) *In any case, there exists $x_* \in M$ satisfying $\|x_* - Tx_*\| < r.$*

For instance, if $X = \ell_2^n$ then (4.4) is satisfied because (4.2) implies

$$J_s(\ell_2^{n'}) = \left(\frac{2n'}{n'+1}\right)^{1/2} < \left(\frac{2n}{n'+1}\right)^{1/2} = J_s(\ell_2^n) \text{ if } n' < n.$$

Therefore, if one of the conditions (i)-(iv) in Theorem 4.5 is fulfilled, then there exists $x_* \in M$ satisfying

$$\|x_* - Tx_*\| \le \frac{1}{2} J_s(\ell_2^n) r = \left(\frac{n}{2(n+1)}\right)^{1/2} r.$$

If $X = \ell_2$ then (4.5) is satisfied because (4.2) implies

$$J_s(\ell_2^n) = \left(\frac{2n}{n+1}\right)^{1/2} < 2^{1/2} = J_s(\ell_2) \quad \text{for any} \quad n \in \mathbb{N}.$$

Hence, if one of the conditions (i)-(iv) in Theorem 4.5 is fulfilled, then there exists $x_* \in M$ satisfying

$$\|x_* - Tx_*\| < \frac{1}{2} J_s(\ell_2) r = 2^{-1/2} r.$$

The invariant degrees given in Theorems 4.4 - 4.5 by using the self-Jung constant should be the best ones which are valid for all r- roughly k- contractive or around r- continuous or uniformly r- continuous self-mappings defined in corresponding normed linear spaces. For instance, consider the n- dimensional Euclidean space $X = \ell_2^n$ and a subset $S = \{x_1, \ldots, x_{n+1}\} \subset \ell_2^n$ of $n+1$ linearly independent points satisfying $\|x_i - x_j\| = r > 0$ for $i \ne j$. Then $\overline{x} = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i$ is the center of the n- dimensional regular simplex $M = \operatorname{conv} S$ and

$$\|x_i - \overline{x}\| = r_M(M) = \frac{1}{2} J_s(\ell_2^n) r = \left(\frac{n}{2(n+1)}\right)^{1/2} r \quad \text{for} \quad 1 \le i \le n+1.$$

By choosing $Tx \in S$ such that $\|x - Tx\| = \max_{s \in S} \|x - s\|$, we have

$$\|Tx - Ty\| \le \operatorname{diam} M = r \quad \text{for all} \quad x, y \in X,$$

i.e., the mapping $T : M \to M$ is r- roughly k- contractive and uniformly r- continuous, and

$$\|\overline{x} - T\overline{x}\| = r_M(M) < \|x - Tx\| \quad \text{for} \quad x \in M \setminus \{\overline{x}\}.$$

Hence, $\gamma = \frac{1}{2} J_s(\ell_2^n) r$ is the smallest invariant degree of T.

To complete this section, let us mention some related results.

A pioneer work was done by Klee [23] who considered self-mappings $T : M \to M$ defined on a compact convex subset M of a normed linear space and satisfying

$$\forall x \in M \ \exists \text{ neighborhood } U_x : \operatorname{diam} T(U_x \cap M) \le r \tag{4.6}$$

and showed that

$$\forall \gamma > r \ \ \exists x_* \in M : \|x_* - Tx_*\| \le \gamma. \tag{4.7}$$

Note that the requirement (4.6) is stronger than the uniform r- continuity.

For a nonempty, compact and convex subset M of a normed linear space, Bula [8] proved that if $T: M \to M$ is uniformly r- continuous then (4.7) holds true, and if

$$\forall x \in M \ \forall \varepsilon > 0 \ \ \exists \delta > 0 : y \in M, \|x - y\| < \delta \Rightarrow \|Tx - Ty\| < r + \varepsilon$$

(i.e., T is 0-r- continuous, according to Section 3) then

$$\forall \gamma > 2r \ \ \exists x_* \in M : \|x_* - Tx_*\| \le \gamma.$$

Note that a 0-r- continuous mapping is around r'- continuous if $r' \ge 2r$.

In [10], Cromme and Diener used two measures of discontinuity defined by

$$\delta(T) := \sup_{x \in M} \limsup_{\sigma \to 0} \sup_{y \in B_\sigma(x)} \|Tx - Ty\|,$$
$$\delta'(T) := \sup_{x \in M} \limsup_{\sigma \to 0} \sup_{y,z \in B_\sigma(x) \setminus \{x\}} \|Ty - Tz\|,$$

where $B_\sigma(x) = \{y \in M \mid \|x - y\| < \sigma\}$, to state the following invariant property of self-mapping $T: M \to M$ on compact and convex subset M of $\mathbb{R}^n$:

$$\exists x_* \in M : \|x_* - Tx_*\| \le \delta(T),$$
$$\forall \gamma > \delta'(T) \ \ \exists x_* \in M : \|x_* - Tx_*\| \le \gamma.$$

This result was improved in [36].

Kirk [21] investigated so-called h-nonexpansive mappings $T: M \to M$ on some metric space (M, d) defined by

$$d(Tx, Ty) \le \max\{d(x, y), h\} \ \text{ for all } \ x, y \in M$$

and received for certain bounded metric spaces M

$$\exists x_* \in M : \|x_* - Tx_*\| \le h$$

and further interesting results.

5. Rough convexity

Recall that a set D of a linear space is called convex if

$$\forall x_0, x_1 \in D \ \forall \lambda \in (0,1) : x_\lambda := (1 - \lambda)x_0 + \lambda x_1 \in D, \tag{5.1}$$

and a function $f: D \to \mathbb{R}$ (on a convex set D) is said to be convex if the Jensen inequality

$$f(x_\lambda) \le (1 - \lambda)f(x_0) + \lambda f(x_1) \tag{5.2}$$

holds true for all $x_0, x_1 \in D$ and $\lambda \in (0,1)$.

Convex functions have many nice properties, for instance:

(P_1) Each lower level set is convex;

(P_2) Each local minimizer is a global minimizer;

(P_3) Each stationary point x_* defined by $0 \in \partial f(x_*)$ is a global minimizer;

(P_4) If a function attains its maximum on a compact convex domain, then it does so at least at one extreme point of this domain.

They also have interesting analytical properties, which will be mentioned in Section 5.3.

In order to obtain similar results for nonconvex functions, there are many concepts of generalized convexities, such as quasiconvexity, pseudoconvexity..., which can be read, for instance, in [3], [16], [17], [24], [52], [57]... The general scheme of these generalizations is

$$f \text{ is } P\text{-convex} \Leftrightarrow (\forall x_0, x_1 \in D: \text{the convexity property } P \text{ is fulfilled in } [x_0, x_1]),$$

where

$$[x_0, x_1] := \{(1-\lambda)x_0 + \lambda x_1 : \lambda \in [0,1]\}.$$

There arise some questions: How often could such a property be satisfied for all $x_0, x_1 \in D$? How many practical nonconvex problems can be covered by such "fine generalizations"? Actually, the image of convexity is sometimes a question of the point of view. Many functions seem to be convex, but they are not, and many nonconvex functions are able to "get convex" by choosing a suitable point of view. If an observer stands far enough away from the object, he cannot recognize disfigurements in small domains, and it seems to be better. Mathematically, for allowing small nonconvex blips, the demand "for all $x_0, x_1 \in D$" is weakened by the requirement "for all $x_0, x_1 \in D$ with $\|x_0 - x_1\| \geq r$", where $r > 0$ denotes the roughness degree. So a "rough generalization" of convex functions has the form

$$f \text{ is } P\text{-convex} \Leftrightarrow (\forall x_0, x_1 \in D \text{ with } \|x_0 - x_1\| \geq r : P \text{ is fulfilled in } [x_0, x_1]).$$

There are several concepts of rough convexities. Let us mention some of them. Note that the Greek letters $\rho, \delta,$ and γ in the following definitions are used as names, not as parameters.

$f : D \subset X \to \mathbb{R}$ (on a convex set D) is said to be ρ-*convex* w.r.t. the roughness degree r provided that (5.2) is fulfilled for all $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| \geq r$ and for all $x_\lambda \in [x_0, x_1]$. This notion was proposed by Klötzler and investigated by Hartwig [16]-[17] and Söllner [56].

According to Hu, Klee, and Larman [19], f is called δ-*convex* w.r.t. the roughness degree r if (5.2) is fulfilled for all $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| \geq r$ for all $x_\lambda \in [x_0, x_1]$ with $\|x_\lambda - x_0\| \geq r/2$ and $\|x_\lambda - x_1\| \geq r/2$. Analytical properties of δ-convex functions were considered in [32].

f is said to be γ-*convex* w.r.t. the roughness degree $r > 0$ provided that $\|x_0 - x_1\| \geq r$ implies

$$f(x_0') + f(x_1') \leq f(x_0) + f(x_1), \tag{5.3}$$

where $x_0', x_1' \in [x_0, x_1]$, $\|x_0 - x_0'\| = r$ and $\|x_1 - x_1'\| = r$. We introduced this notion in [29] and [31], and investigated it further in [46] and [47].

Relations between six kinds of roughly convex functions and their basic properties were stated in [33].

5.1. Some aspects of γ- convex functions

To understand the idea of γ- convexity, let us consider real-valued functions on the real line. It is well known that a convex function $f : \mathbb{R} \to \mathbb{R}$ is characterized by the following slope property: If $x_1 < x_2$, $\gamma_1 > 0$, $\gamma_2 > 0$, and $x_1 + \gamma_1 \le x_2 + \gamma_2$, then

$$\frac{f(x_1 + \gamma_1) - f(x_1)}{\gamma_1} \le \frac{f(x_2 + \gamma_2) - f(x_2)}{\gamma_2}.$$

For our generalization, we require this slope property to be maintained for $\gamma_1 = \gamma_2 = r$, where r is a given positive number, i.e.,

$$h(x) = \frac{1}{r}\big(f(x + r) - f(x)\big) \text{ is nondecreasing,} \tag{5.4}$$

which is just equivalent to (5.3) when $D = \mathbb{R}$. For continuous f, this means that

$$F(x) = \frac{1}{r}\int_x^{x+r} f(t)dt \quad \text{is convex}$$

since $F'(x) = h(x)$, i.e., f is "convex in average". The latter one is actually our original idea of γ- convexity. But, for functions on multidimensional spaces, it is difficult to handle with such average convexity. Therefore, we use the monotony (5.4) for our generalization.

Proposition 5.1. ([29], [31])

(a) *If $f : D \subset X \to \mathbb{R}$ is convex then it is γ- convex w.r.t. any roughness degree $r > 0$.*

(b) *If $f : \mathbb{R} \to \mathbb{R}$ is periodic with period r then it is γ- convex w.r.t. the roughness degree r.*

(c) *If f_1 and f_2 are γ- convex w.r.t. the roughness degree r and if $\lambda_1 \ge 0$ and $\lambda_2 \ge 0$ then $\lambda_1 f_1 + \lambda_2 f_2$ is γ- convex w.r.t. r.*

For instance, for an arbitrary positive rational number r, the function

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases} \tag{5.5}$$

is periodic with period r, hence, it is γ- convex w.r.t. the roughness degree r. In fact, in this theory, periodic functions play the role of constant ones, that is of value because many processes in nature are periodic or contain periodic components.

Another example is function $f(x) = x^2 + k\sin x,$ which is γ- convex w.r.t. the roughness degree $r = 2\pi$ because x^2 is convex and $k\sin x$ is periodic with period $r = 2\pi.$

It is easy to verify that if $f : X \to \mathbb{R}$ is additive, i.e., if $f(x+y) = f(x) + f(y)$ for all $x,$ $y \in X,$ then f is γ- convex w.r.t. any roughness degree $r > 0.$

We see that the class of γ- convex functions is relatively large. The next property emphasizes this aspect once again.

Proposition 5.2. ([31]) *Suppose that* $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *is of bounded variation. Then, for every positive constant* $r,$ *there exist two functions* f_1 *and* f_2 *which are* γ- *convex w.r.t. the roughness degree* r *such that* $f = f_1 - f_2.$

The previous property corresponds to the fact that continuous functions can be approximated by the difference of convex functions, which is useful for global optimization (see [18]). An important aspect of the γ- convexity is its compatibility with numerical computation. Since only finitely many states can be represented by digital computers, a continuous function $g : \mathbb{R} \to \mathbb{R}$ must be approximated by a piecewise constant function $f : \mathbb{R} \to \mathbb{R}.$ After standardization, these functions satisfy

$$f(x) = f([x]) = g([x]) \quad \text{for all} \quad x \in \mathbb{R}, \tag{5.6}$$

where $[x]$ denotes the integer part of $x.$ In general, f is not convex although g is convex. In contrast, γ- convexity can be maintained by this approximation.

Proposition 5.3. ([48]) *Suppose* (5.6). *Then* f *is* γ- *convex w.r.t. the roughness degree* $r = 1$ *if and only if* g *is* γ- *convex w.r.t. the roughness degree* $r = 1.$

In particular, if g is convex then f is γ- convex w.r.t. the roughness degree $r = 1.$ Moreover, we showed in [48] that ρ- convexity and δ- convexity do not have this advantage. Thus, for computer world, convex functions usually appear as γ- convex ones.

A typical difficulty often appears when doing with generalized convex functions is the lacking of a suitable sufficient condition for the corresponding generalized convexity. From this point of view, γ- convexity is an exception. Similarly to the classical convexity, we have the following sufficient condition.

Proposition 5.4. ([46], [47]) *Suppose that* $D \subset X$ *is an open convex subset and* $f : D \to \mathbb{R}$ *is Gateaux differentiable at every point of* $D,$ *and* $z \mapsto \langle f'(z), x_1 - x_0 \rangle$ *is a continuous mapping of any interval* $[x_0, x_1] \subset D$ *into* $\mathbb{R}.$ *Then* f *is* γ- *convex if and only if its Gateaux derivative* f' *is* γ- *monotone in the following sense:*

$$\text{if } x, y \in D, \ \|x - y\| = \gamma \ \text{ then } \langle f'(y) - f'(x), x - y \rangle \geq 0.$$

Applying the one-dimensional version ([46]) of the above result, Kripfganz showed in [25] that Favard's 'fonction penetrante' denoting the optimal value function of a geometrical parametric optimization problem is γ- convex.

5.2. Optimization properties

A subset $M \subseteq X$ is said to be *outer* γ- *convex* w.r.t. the roughness degree $r > 0$ if for all x_0 and x_1 in M there exist $k \in \mathbb{N}$ and

$$\begin{aligned} &\lambda_i \in [0,1], \;\; i = 0,1,\ldots,k, \quad \text{with} \quad \lambda_0 = 0, \, \lambda_k = 1, \\ &0 \le \lambda_{i+1} - \lambda_i \le \frac{r}{\|x_0 - x_1\|} \;\; \text{for} \;\; i = 0,1,\ldots,k-1, \end{aligned} \tag{5.7}$$

such that

$$x_{\lambda_i} \in M \quad \text{for} \quad i = 1,2,\ldots,k-1, \tag{5.8}$$

where $x_\lambda = (1-\lambda)x_0 + \lambda x_1$. (5.7)-(5.8) imply $\left\|x_{\lambda_{i+1}} - x_{\lambda_i}\right\| \le r$ for $i = 0,1,\ldots,k-1$.

A function $f : D \to \mathbb{R}$ defined on some convex $D \subset X$ is called *outer* γ- *convex* w.r.t. the roughness degree $r > 0$ if for all x_0 and x_1 in D there exist $k \in \mathbb{N}$ and $\lambda_i \in [0,1]$, $i = 0,1,\ldots,k$, satisfying (5.7) such that

$$f(x_{\lambda_i}) \le (1-\lambda_i)f(x_0) + \lambda_i f(x_1) \text{ for } i = 1,2,\ldots,k-1.$$

Proposition 5.5. ([44]) *Suppose that* $f : D \subset X \to \mathbb{R}$ *is outer* γ- *convex w.r.t. the roughness degree* $r > 0$. *Then:*

(P_1^r) *Each lower level set* $\mathcal{L}_\alpha(f) = \{x \in D : f(x) \le \alpha\}$ *is outer* γ- *convex w.r.t.* r.

(P_2^r) *Each* r- *local minimizer* x_* *defined by*

$$f(x_*) \le f(x) \text{ for all } x \in D \text{ satisfying } \|x - x_*\| \le r$$

is a global minimizer.

It is easy to recognize that (P_1^r) and (P_2^r) are rough versions of (P_1) and (P_2), respectively. Since ρ-, δ-, and γ- convex functions are outer γ- convex w.r.t. corresponding roughness degree, they also have these properties. Concretely, if a function is ρ- convex or δ- convex or γ- convex w.r.t. the roughness degree $r > 0$ then it fulfils (P_2^r). Similar or even stronger results are given in [19], [31], and [33].

Let us come now to a modification of property (P_3), which actually says that, for convex functions, the necessary condition $0 \in \partial f(x_*)$ for (local) minimizers becomes a sufficient condition for global minimizers. To state a rough version of (P_3), we use the so-called γ- *subdifferential* (w.r.t. the roughness degree $r > 0$) of $f : X \to \mathbb{R}$ defined by

$$\partial_r f(x) := \{\xi \in X^* : \text{ for all } s \in X \text{ satisfying } \|s\| = r \text{ there exist } x', x'' \in X$$
$$\text{such that } x' - x'' = s,\ x \in [x', x''],\ \langle \xi, s \rangle \le f(x') - f(x'')\},$$

which was introduced and investigated in [29] and [31]. Note that the more general restricted case $f : D \subset X \to \mathbb{R}$ was considered there, but, for the sake of simplicity, we only mention the unrestricted case $D = X$ here. In general, $\partial_r f(x)$ is convex for any $x \in X$. If $\dim X < +\infty$ and f is continuous on $\overline{B}_r(x) = \{x' \in X : \|x' - x\| \le r\}$ then $\partial_r f(x)$ is compact. It was shown in [13] that the γ-subdifferential $\partial_r f(x)$ could be empty, even if f is γ-convex. But, the γ-subdifferential of any function at a global minimizer is nonempty, as given in the following.

Proposition 5.6. ([31]) *Let* $f : X \to \mathbb{R}$. *If*

$$f(x_*) \le f(x) \textit{ for all } x \in D \textit{ satisfying } \|x - x_*\| \le r$$

then $0 \in \partial_r f(x_*)$.

This is a useful necessary condition for characterizing global minimizers, because many local minimizers, which are not a global minimizer, do not satisfy it. For instance, for very large $k > 0$, the number of local minimizers of function $f(x) = x^2 + k \sin x$ is also very large, which tends to $+\infty$ as $k \to +\infty$, while only one of them is a global minimizer. In such a case, it is expensive to use the classical derivative alone to determine global minimizers. On the contrary, for a continuous function $f : \mathbb{R} \to \mathbb{R}$, the necessary condition $0 \in \partial_r f(x_*)$ means

$$x_* \in [y, y + r] \text{ for some } y \in \mathbb{R} \text{ satisfying } f(y) - f(y + r) = 0,$$

which has exactly one solution $y \in \mathbb{R}$ if $f(x) = x^2 + k \sin x$ and $r > \left|k \sin \frac{r}{2}\right|$. In particular, for $r = 2\pi$, $y = -\pi$ is the only solution of equation $f(y) - f(y + r) = 0$, and our necessary condition is $x_* \in [-\pi, \pi]$. It is already quite helpful because this function only has one local minimizer in $[-\pi, \pi]$, which can be determined uniquely by the necessary conditions $f'(x_*) = 0$ and $f''(x_*) \ge 0$ (see [31] for more details).

In the previous example, many points z satisfy the roughly stationary condition $0 \in \partial_r f(z)$, which indicates that such an r-stationary point is not necessarily a global minimizer. Thus (P_3) may be modified as follows:

(P_3^r) If $0 \in \partial_r f(z)$ then $\inf_{x \in \overline{B}_r(z)} f(x) = \inf_{x \in X} f(x)$.

Proposition 5.7. ([31]) *Suppose that* $f : X \to \mathbb{R}$ *is* γ-*convex w.r.t. the roughness degree* $r > 0$ *and* $0 \in \partial_r f(z)$. *Then, for all* $y \in X$ *there exists* $y' \in \overline{B}_r(z)$ *such that* $f(y') \le f(y)$, *which implies immediately* (P_3^r). *If* f *attains its global minimum, then it has a global minimizer in* $\overline{B}_r(z)$.

To obtain a modification of (P_4) for roughly convex functions, we use the following notion: For some given $r > 0$ and a convex set $D,$ $z \in D$ is said to be an r- *extreme* point of D provided

$$\text{if } z = \tfrac{1}{2}(x_0 + x_1) \text{ for some } x_0, x_1 \in D \text{ then } \|z - x_0\| = \|z - x_1\| < r.$$

This notion was introduced in [30] for representing bounded convex sets which may be nonclosed.[1]

(P_4^r) If a function attains its maximum on a bounded convex domain, then it does so at least at an r- extreme point of this domain.

Note that the domain mentioned in (P_4^r) does not need to be compact.

Proposition 5.8. ([33]) *Suppose that* X *is a pre-Hilbert space and* $f : X \to \mathbb{R}$ *is midpoint* γ- *convex w.r.t. the roughness degree* $r > 0,$ *that is defined by*

$$\textit{the Jensen inequality } (5.2) \textit{ is fulfilled for } \lambda = \tfrac{1}{2} \textit{ whenever } \|x_0 + x_1\| = 2r.$$

Then (P_4^r) *holds true for this* $f.$

If a function is ρ- convex or δ- convex or γ- convex w.r.t. the roughness degree $r > 0$ then it is midpoint γ- convex w.r.t. $r.$ Therefore, the previous assertion is also valid for such a function.

5.3. Analytical properties

Convex functions have numerous interesting analytical properties. For instance, a proper convex function f on $\mathbb{R}^n$ is continuous relative to $\operatorname{ri}(\operatorname{dom} f),$ Lipschitzian relative to any closed bounded subset of $\operatorname{ri}(\operatorname{dom} f),$ and almost everywhere differentiable in $\operatorname{int}(\operatorname{dom} f)$ (see [52], [53]...). These properties often fail to be true for generalized convex functions when the Jensen inequality (5.2) is not satisfied for all $x_0, x_1 \in D$ and $\lambda \in [0,1]$ as required in (5.1), especially for roughly convex functions where only x_0 and $x_1 \in D$ satisfying $\|x_0 - x_1\| \geq r > 0$ are taken into consideration.

Some analytical properties of ρ-convex functions were investigated in [16] and [56]. In general, there are discontinuous ρ-convex functions even if $D \subset X = \mathbb{R}$ (see [33]). But a

[1] Minkowski showed $C = \operatorname{conv}(\operatorname{ext} C)$ if C is a compact convex subset of a finite dimensional real linear space, where $\operatorname{ext} C$ denotes the set of extreme points of $C.$ By using closed convex hull instead of convex hull, Krein and Milman extended this result for compact convex subsets in locally convex Hausdorff linear spaces. Klee generalized these results for locally compact closed convex subsets which contain no line. To have a similar representation for convex subsets of a finite dimensional linear metric space, which are bounded but not necessarily closed, we replaced $\operatorname{ext} C$ by $\operatorname{ext}_r C,$ where $\operatorname{ext}_r C$ denotes the set of r -extreme points of $C.$

ρ-convex function $f : D \subset \mathbb{R} \to \mathbb{R}$ w.r.t. the roughness degree $r > 0$ is continuous at any x satisfying $[x - r, x + r] \subset D$, and if D contains such an interval $[x - r, x + r]$ then this f is locally bounded.

The boundedness and the continuity of δ-convex functions on the real line were studied in [32]. If $f : D \subset \mathbb{R} \to \mathbb{R}$ is δ-convex w.r.t. the roughness degree $r > 0$ then it is bounded from above in each closed bounded interval of $D^r = \{x \in D : [x - r/2, x + r/2] \subset D\}$ and bounded from below in each bounded interval of $D_r = \{x \in D : x - r \in D \text{ or } x + r \in D\}$. In particular, if $D = \mathbb{R}$ then f is locally bounded. But there are totally discontinuous δ-convex functions even if $D = \mathbb{R}$. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = a^{[|x| + 0.5]} + \phi(x)$$

where ϕ is given in (5.5), is totally discontinuous, and it is δ-convex w.r.t. the roughness degree $r = 2$ if and only if $a \geq (3 + \sqrt{13})/2$.

Since γ-convexity is weaker than δ-convexity, there are totally discontinuous γ-convex functions on the real line. In fact, the function ϕ defined in (5.5) is such an example, as already mentioned after Proposition 5.1. Another example was considered in [33], namely the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} 1 & \text{if } x = 0, \\ q & \text{if } x = p/q \text{ for } p \in \mathbb{Z},\ q \in \mathbb{N},\ (p, q) = 1, \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

where (p, q) denotes the greatest common divisor of the absolute values $|p|$ and $|q|$ of the integers p and q, respectively. Both functions φ and $-\varphi$ are periodic with period $r = 1$, therefore they are γ-convex w.r.t. the roughness degree $r = 1$. Obviously, they are totally discontinuous. Moreover, they are totally unbounded on the entire real line because

$$\limsup_{x' \to x} \varphi(x') = +\infty \text{ and } \liminf_{x' \to x} -\varphi(x') = -\infty \text{ for all } x \in \mathbb{R}.$$

Apparently, γ-convex functions do not surely have any good analytical properties. But the contrary was shown in [46]: γ-convex functions on the real line have so-called *conservation properties*, i.e., positive properties which remain true on every bounded closed interval or even in the entire domain, if they hold in a certain closed interval with length equal to the roughness degree.

Theorem 5.9. ([46]) *Suppose that* $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *is* γ-*convex w.r.t. the roughness degree* $r > 0$, *and that* $[c, c + r]$ *is some subinterval of* D.

(a) *If* f *is bounded (from above or/and from below), or of bounded variation, or Riemann- or Lebesgue-integrable in the interval* $[c, c + r]$, *then it has this property on every bounded subinterval of* D.

(b) *If f is continuous a.e. or differentiable a.e. on the interval $[c, c+r]$, then it has this property on the entire domain D.*

Conversely, γ- convex functions on the real line possess so-called *infection properties* being negative ones which "propagate" to other places once appearing somewhere.

Proposition 5.10. ([46]) *Suppose that $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is γ- convex with the roughness degree $r > 0$. If there exists a subinterval of D in which f is unbounded (from below or/ and from above), or of unbounded variation, or nowhere differentiable, or not integrable, or it has uncountable many discontinuities, then it has this property on any subinterval $[x, x+r]$ of its domain. If f is discontinuous almost everywhere on a certain interval $[c, c+r] \subset D$, so is it on every nonempty open subinterval of D.*

In this context, a closed interval with length r (i.e., a closed ball with radius $r/2$) plays the role of an elementary cell which contains positive and negative characteristics of the considered object. This corresponds to the phenomenon that information about an organism can be read in its genes.

Analytical properties of γ- convex functions in normed linear spaces were studied in [47].

To obtain more positive analytical properties, we investigated in [14] and [15] a special class of γ- convex functions which fulfil the Jensen inequality (5.2) at both points $x_0', x_1' \in [x_0, x_1]$ satisfying $\|x_0 - x_0'\| = r$ and $\|x_1 - x_1'\| = r$ whenever the distance between x_0 and $x_1 \in D$ is greater than the roughness degree $r > 0$, thus, they are called *symmetrically γ- convex*.

5.4. Additional comments

Many functions appearing in economical problems are considered to be convex or concave, in the classical sense or some generalized sense. But this ideal assumption often fails if real jumps are taken into account. For instance, consider the transportation cost function f of a company using identical vehicles, then f is not continuous everywhere but has jumps at $nr, n \in \mathbb{N}$, where r denotes the maximal load of one vehicle, because an extra cost of driver, car, and fuel is caused if an additional vehicle is used although it carries only a little good amount. Therefore, this function f cannot be convex or concave, but it could be roughly convex (or roughly concave) w.r.t. the roughness degree r. Such a practical example of roughly convex functions was investigated in [45] to illustrate the practical relevance of the concept of rough convexities.

From application point of view, a generalized convexity is more suitable if it is stable, at least by linear disturbances. We say that a function f is stable w.r.t. some property (P) if (P) remains true when f is disturbed by addition with an arbitrary linear functional ξ with sufficiently small norm $\|\xi\|$. In [42], we showed that typical generalized convexities, such as quasiconvexity, explicit quasiconvexity, and pseudoconvexity, are not stable w.r.t. (P_1), or (P_2), or (P_3), respectively, that is the property which the corresponding generalized convexity must maintain. Hence, a stable concept called s- *quasiconvexity* was introduced and studied there.

A function f is said to be *absolutely stable* w.r.t. some property (P) if $f+\xi$ fulfils (P) for any linear functional ξ. This condition is actually very strong. In fact, if a lower semicontinuous function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is absolutely stable w.r.t. property (P_1) then it must be convex. This and some other results on the absolute stability of generalized convexities were shown in [43] and [44]. In particular, our concepts of rough convexities are absolutely stable w.r.t. the optimization properties which we like to keep by generalization.

Up to now, there are little investigations on roughly convex sets. Some properties of outer γ- convex sets are stated in [39] and [44]. For instance, if a closed subset S of some normed linear space X is outer γ- convex w.r.t. the roughness degree $r>0$ then $S\subset S+\overline{B}_\sigma(0)$ for $\sigma=\frac{1}{2}J_s(X)r$. Consequently, if $x\in X$ satisfies $\overline{B}_{\sigma'}(x)\cap S=\emptyset$ for some $\sigma'>\frac{1}{2}J_s(X)r$ then $x\notin \mathrm{cl}(\mathrm{conv}\, S)$, hence, x and S can be strictly separated by a nontrivial continuous linear functional.

6. Concluding remarks

A crucial key for scientific research is to choose an appropriate reference point or a suitable wise for consideration so that interested objects appear simpler and more reasonable, and some related laws can arise or show their face.

Consider, for instance, a Lebesgue integrable function $f:D\subset\mathbb{R}^n\to\mathbb{R}$. In general, it could be so wild and so complicated that it is hard to determine its essential supremum (denoted by $\mathrm{ess\,sup}\, f$), while this problem is closely related to that of calculating the norm in L_∞. But by using the auxiliary function $F(\alpha)=\int_{\{x\in D: f(x)\geq\alpha\}}(f(x)-\alpha)d\mu,$ we have to do with a very nice function $F:\mathbb{R}\to\mathbb{R},$ which is continuous, nonnegative, nonincreasing, convex, and has almost everywhere the derivative $F'(\alpha)=-\mu\{x\in D: f(x)\geq\alpha\}$. Moreover, $\mathrm{ess\,sup}\, f$ is equal to the first root $\sup\{\alpha\in\mathbb{R}: F(\alpha)>0\}$ of $F,$ which can be calculated by the Newton method. This fact was used in [49] as an approach to an integral method for finding global maxima.

Further similar positive experiences are mentioned in this paper, such as the rough continuity of linear operators mapping an arbitrary normed space into a finite dimensional normed space (Theorem 3.2), the rough lower semicontinuity of convex functions defined on a convex set of any normed space (Theorem 3.3), and conservation properties of γ- convex functions (Theorem 5.9), which are not attainable in the classical fine context. An essential factor for the mentioned advantage is that a point x is not considered separately but embedded into its r- neighborhood $\overline{B}_r(x)$. To understand this effect better, let us use the following comparison example: If we treat our world as a set of single atoms, then it seems to be too complicated to assert something more than physical laws at atom level. But if atoms are collected in suitable units, namely people, i.e., the atom set appears as mankind, then the image of our world is much more simple and more beautiful, and "suddenly" it is possible to discuss a lot of properties, such as beauty, talent, psychology, habit, health, and genetics, which are meaningless in the context of an atom set. Back to our theory: in the rough analysis, closed balls with radius equal to some positive roughness degree are suitable packing units which help to make our objects simpler so that we are able to discover some interesting properties.

A remarkable feature is that, by working with a positive roughness degree $(r > 0)$, we are able to avoid the closedness assumption in many assertions, for instance, in Proposition 2.11, Theorem 3.4, Theorem 4.5, and Proposition 5.8, although this assumption is necessary for corresponding statements in the classical analysis. It is no accidental outcome at all but a result of our intended effort to develop a suitable approach to such worlds which are rough and possibly open in some respects.

Our paper only presents some beginning results. There are still many to do in this promising research direction, which requires enthusiastic interest, fresh strength, and intensive cooperation.

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