High Dimensional Multilevel Smolyak Approximation

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Abstract: We analyze convergence rates of multilevel Smolyak interpolation/integration for parametric maps $u : U \to X$ taking values in a Banach space X and defined on the parameter domain $U = [-1, 1]^{\mathbb{N}}$. A concrete algorithm is proposed to apriori identify sets of active multiindices and approximation levels for numerical solvers. As an application, we concentrate on the approximation of solutions to parametric linear and nonlinear partial differential equations (PDEs), by using Galerkin methods to approximate the PDE solution at a given point in the parameter domain.

It is well-known, that so-called $(\mathbf{b}_0, \varepsilon)$-holomorphic functions enjoy sparsity properties, which are quantified by the summability of their Taylor polynomial chaos coefficients. With $\mathbf{b}_0 \in (0,\infty)^{\mathbb{N}}$, this holomorphy condition reads: the map $\mathbf{y} = (y_j)_{j\in\mathbb{N}} \mapsto u(\mathbf{y})$ admits a uniformly bounded extension to $O_{\mathbf{b}_0} \subseteq \mathbb{C}^{\mathbb{N}}$, and the extension is holomorphic in each y_j. Here $O_{\mathbf{b}_0}$ denotes a union of certain polydiscs whose radii behave reciprocal to the entries of $\mathbf{b}_0$. With appropriate conforming trial spaces $X_l \subseteq X$, $l \in \mathbb{N}$, it is shown that the parametric Galerkin approximation $\mathbf{y} \mapsto u_l(\mathbf{y}) \in X_l$ possesses the same holomorphy domain $O_{\mathbf{b}_0}$ as the solution u. In addition we assume the existence of $\mathbf{b}_1 \in (0,\infty)^{\mathbb{N}}$ and $\alpha > 0$ such that $\sup_{\mathbf{y}\in O_{\mathbf{b}_1}} \|u(\mathbf{y})\|_Y < \infty$ for a space $Y \hookrightarrow X$ satisfying $\sup_{\|v\|_Y\le 1} \inf_{w\in X_l} \|v - w\|_X = O(l^{-\alpha})$ as $l \to \infty$. Under these hypotheses and for $\mathbf{b}_i \in \ell^{p_i}$, $i \in \{0, 1\}$ such that $0 \le p_0 \le p_1 < 1$, we prove uniform convergence of the multilevel interpolant on U at a rate depending on p_0, p_1 and α. Moreover, computation of the Bochner integral $\int_U u(\mathbf{y}) \bigotimes_{j\in\mathbb{N}} \frac{\mathrm{d}y_j}{2}$ is considered. We investigate a multilevel Smolyak quadrature, which exploits certain cancellation properties and thus allows for improved results exceeding best N-term rates. Up to any $\delta > 0$, all mentioned rates are established in terms of the total computational complexity. A refined analysis of a model integrand class shows that a generally large preasymptotic range may preclude reaching the asymptotic rate for practically relevant numbers of interpolation/quadrature points.

References

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