On the Regularity and the Singular Support of the Minimum Time Function with Hörmander Vector Fields

P. Albano[1], P. Cannarsa[2], and T. Scarinci[3]

Abstract: Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary, Γ, and let $X_1, \dots, X_N$ be smooth real vector fields on an open set $\Omega' \subset \Omega$. We assume that they satisfy the Hörmander bracket generating condition, i.e., $Lie\{X_1, \dots, X_N\}(x) = \mathbb{R}^n$, $\forall x \in \Omega'$. Here, $Lie\{X_1, \dots, X_N\}(x)$ denotes the space of all values at x of the vector fields of the Lie algebra generated by $\{X_1, \dots, X_N\}$. In this context we consider the following Dirichlet problem

$$\begin{cases} \sum_{j=1}^N (X_j T)^2(x) = 1, & x \in \Omega, \\ T(x) = 0, & x \in \Gamma. \end{cases} \tag{1}$$

Existence and uniqueness of the viscosity solution (1) are well-known. Moreover, this solution T is the value function of the time-optimal control problem with target Γ and state equation

$$y'(t) = \sum_{j=1}^N u_j(t) X_j(y(t)), \quad t \geq 0, \quad y(0) = x. \tag{2}$$

The controls $u = (u_1, \dots, u_N)$ take values in the n-dimensional closed ball of unit radius centered at the origin. The quadratic form associated with the eikonal equation (1) is not positive definite. Thus, *singular trajectories* may occur, destroying the smoothness of T.

In this talk, we investigate the regularity of T, the properties of its singular support, and the role played by the singular trajectories. Our main result claims that the singular support of T, which consists of all points at which this function is not Lipschitz, has null Lebesgue measure. Finally, we discuss the relation between our results and the so-called "Sard conjectures" (see L. Rifford and E. Trélat, *Morse-Sard type results in sub-Riemannian geometry*, Math. Ann. 332 (2005)).

The results presented appear in the following preprints: P. Albano, P. Cannarsa, T. Scarinci. *Regularity results for the minimum time function with Hörmander vector fields* (2017), and P. Albano, P. Cannarsa, T. Scarinci. *On the partial regularity of the solution of the subelliptic eikonal equation* (2017).

[1] Department of Mathematics, University of Bologna
Piazza di Porta S. Donato 5, 40127, Bologna, Italy
paolo.albano@unibo.it

[2] Department of Mathematics, University of Rome Tor Vergata
Via della Ricerca Scientifica 1, 00133, Rome, Italy
cannarsa@mat.uniroma2.it

[3] Department of Statistics and Operations Research, University of Vienna
Oskar-Morgenstern-Platz 1, 1090, Wien, Austria
teresa.scarinci@univie.ac.at