What Can We Expect? Computable Upper Bounds to Machine Learning in Inverse Problems Using MCMC

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Abstract: Deep learning based methods have recently given state-of-the art results in several fields of inverse problems, including computed tomography (CT), Photo-acoustic tomography (PAT) and Magnetic Resonance Imaging (MRI). We study inverse problems of the form

$$g = T(f) + \delta g$$

where $T : X \to Y$ is a forward model, $f \in X$ is the signal we want to reconstruct, $g \in Y$ is some measured data and $\delta g \in Y$ is noise.

Machine learning based methods for the solution of these problems typically assume that training data is given in the form of a random variable (f, g). For example, in computed tomography f could be high quality CT scans, considered to be the ground truth, and g could be corresponding low dose CT sinograms. Machine learning would then be done by specifying a parametrized operator T_Θ^{-1} such that $T_\Theta^{-1}(\mathsf{g}) \approx \mathsf{f}$ for f and g in the training data. This can be done by selecting the parameters Θ such that they minimize the mean error

$$\Theta^* \in \arg\min_\Theta E_{(\mathsf{f},\mathsf{g})} \|T_\Theta^{-1}(\mathsf{g}) - \mathsf{f}\|_2^2.$$

We show that under weak assumptions, the performance of T_Θ^{-1} is bounded by the performance of the conditional expectation

$$E_{(\mathsf{f},\mathsf{g})} \|E_{\mathsf{f}}(\mathsf{f}|\mathsf{g}) - \mathsf{f}\|_2^2 \leq \min_\Theta E_{(\mathsf{f},\mathsf{g})} \|T_\Theta^{-1}(\mathsf{g}) - \mathsf{f}\|_2^2.$$

Further, we show that this is a *strict* bound by proving that this can be approximated arbitrarily well by a neural network.

Finally, we compute this lower bound using Markov Chain Monte Carlo (MCMC) for some choices of training data in the setting of CT and compare the results to current state of the art inversion methods based on deep learning and discuss implications to further research into machine learning for inverse problems.

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