Applying a Patched Mesh Method to Efficiently Solve a Singularly Perturbed Reaction-diffusion Problem

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Abstract: We consider the numerical solution of a singularly perturbed reaction-diffusion problem in two dimensions:

$$-\varepsilon^2\Delta u + b(x,y)u = f(x,y), \quad \Omega = (0,1)^2, \quad u = g, \ \partial\Omega,$$

where ε is a *perturbation parameter*, $0 < \varepsilon \ll 1$, and g, b and f are some given functions with $b > 0$. These problems come from a wide variety of mathematical models, from semi-conductor device modelling to computational finance. Due to the parameter ε, which may be arbitrarily small, the solution exhibits corner and edge layers. Therefore, classical numerical methods have difficulties generating solutions that are accurate for all ε, and so special layer-resolving meshes, such as the piecewise uniform grid of Shishkin type, are required [1]. The mesh-widths of the Shishkin grid are of very different orders of magnitudes in different regions: $\mathcal{O}(\varepsilon N^{-1}\ln N)$ near the boundaries, and $\mathcal{O}(N^{-1})$ in the interior, where N is the discretisation parameter. In [3], it is shown that it can be problematic to apply direct solvers to this discretisation. Instead, a specialised multigrid-based *boundary layer preconditioner* (for conjugate gradients) is constructed. This involves using different preconditioning techniques in each of the corner, edge, and interior regions. Although successful, this approach can be difficult to implement and analyse.

We propose a simpler method, based on a patched mesh method [2], which allows us construct a solution based on solving associated, but independent, problems on separate subregions. In these subregions, the mesh is locally uniform, greatly simplifying the theoretical and practical considerations of applying bespoke solution strategies (including multigrid methods) in each region, and makes parallelization of the process relatively straight-forward.

1. C. Clavero, J.L. Gracia and E. O'Riordan, *A parameter robust numerical method for a two dimensional reaction-diffusion problem*, Math. Comp., 74 (2005), pp. 1743-1758.

2. C. de Falco and E. O'Riordan, *A patched mesh method for singularly perturbed reaction-diffusion equations*. In Hegarty, A.F.; Kopteva, N.; O' Riordan, E.; Stynes, M. (Eds.), Proc. BAIL 2008, Vol. 69 of Lect. Notes Comput. Sci. Eng. p117-127 (2009).

3. S. MacLachlan and N. Madden, *Robust solution of singularly perturbed problems using multigrid methods*, SIAM J. Sci. Comput., 35 (2013), pp. A2225-A2254.

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