On Maximization of a Psd Quadratic Form of Low Rank over a Box by Incremental Enumeration of Vertices of a Zonotope and Polynomial-time Approximations of the Objective Value

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Abstract: We consider the problem $\max_{x\in\mathbb{R}^n} x^TAx$ s.t. $x \in C := \{\xi : \underline{x} \leq \xi \leq \overline{x}\}$, where $\underline{x}, \overline{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is psd with rank $d \ll n$. This problem can be rewritten into the form $\max_{y\in\mathbb{R}^d} y^Ty$ s.t. $y \in Z$, where Z is the generator-presented zonotope $\{G\xi : \underline{x} \leq \xi \leq \overline{x}\}$ and $G \in \mathbb{R}^{d\times n}$ is such that $G^TG = A$. This reformulation converts the enumeration of 2^n vertices of the cube C into the problem of enumeration of vertices of the d-dimensional zonotope Z, which has a "much lower" number of vertices compared to 2^n. In particular, when $d = O(1)$, the number of vertices of Z is $O(n^{d-1})$. We employ two versions of the recent IncEnu algorithm for enumeration of vertices of Z and compare them with Avis-Fukuda's Reverse Search method. Then we discuss a method for approximation of the optimal value of the problem based on "geometric rounding" of Z by a pair of Löwner-John ellipsoids (using Goffin's algorithm adapted for generator-presented zonotopes) over which the function x^TAx can be maximized efficiently. We also discuss questions related to the tightness approximation and complexity-theoretic considerations of the problem.

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