Convex Programming with SOS-Convex Polynomials

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Abstract: Consider the convex optimization problem

$$\mathrm{P}: \quad f^* := \inf_x \{f(x) \;:\; g_j(x) \leq 0, \quad j = 1, \ldots, m\} \tag{1}$$

where $f, g_j \in \mathbf{R}[x]$ $(= \mathbf{R}[x_1, \cdots, x_n])$ are given *convex* polynomials. Of course, problem (1) can be solved by interior point methods, e.g. using the logartithmic barrier function $x \mapsto \phi(\mu, x) := \mu^{-1} f(x) - \sum_{j=1}^m \ln(-g_j(x))$ with barrier parameter μ. However in general, ϕ does not possess the higly desirable *self-concordance* property.

On the other hand, consider the subclass of problems (1) whose data f, g_j are SOS-convex polynomials. A polynomial $h \in \mathbf{R}[x]$ is SOS-convex if its Hessian $\nabla^2 h$ is a sum of squares, i.e., $\nabla^2 h = QQ^T$ for some matrix polynomial $Q \in \mathbf{R}[x]^{n \times s}$ and some $s \in \mathbb{N}$; see [1]. Importantly, SOS-convexity can be checked numerically by solving a semidefinite program. Next, given a sequence $\mathbf{y} = (y_\alpha)$ indexed in the canonical basis $(X^\alpha)_{\alpha \in \mathbf{N}^n}$ of $\mathbf{R}[x]$, let $L_\mathbf{y} : \mathbf{R}[x] \to \mathbf{R}$ be the linear mappping

$$f \; (= \sum_\alpha f_\alpha \, x^\alpha) \in \mathbf{R}[x] \;\longmapsto\; L_\mathbf{y}(f) \;=\; \sum_\alpha f_\alpha \, y^\alpha.$$

• We show that if (a) Slater's condition holds for P and (b) f, g_j are SOS-convex, $j = 1, \ldots, m$, then f^* is also the optimal value of the semidefinite program

$$\left\{ \begin{array}{ll} \inf_\mathbf{y} & L_\mathbf{y}(f) \\ \text{s.t.} & M_d(\mathbf{y}) \succeq 0, \quad y_0 = 1, \quad L_\mathbf{y}(g_j) \geq 0, \; j = 1, \ldots, m \end{array} \right. , \tag{2}$$

where $M_d(\mathbf{y})$ is the moment matrix associated with $\mathbf{y}$, with rows and columns indexed in the canonical basis (X^α), and defined by:

$$M_d(\mathbf{y})(\alpha, \beta) \;=\; L_\mathbf{y}(X^{\alpha+\beta}) \;=\; y_{\alpha+\beta}, \qquad \alpha, \beta \in \mathbf{N}^n; \quad |\alpha|, \, |\beta| \leq d,$$

with $2d \geq \max[\deg f, \max_j[\deg g_j]]$.

• Hence, in contrast with (1), one may solve the semidefinite program (2) via interior point methods and a self-concordant logarithmic barrier function, as e.g. in several free access SDP solvers. In addition, the semidefinite constraint $M_d(\mathbf{y}) \succeq 0$ of (2) does *not* depend on the original problem data $\{f, g_j\}$, a very nice property that could be exploited in a specialized SDP solver dedicated to problems (1) with SOS-convex polynomials.

[1] J. W. Helton and J. Nie, Semidefinite representation of convex sets, Math. Prog., to appear.

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