# Variational Inequalities for the MHD Systems and Application to Controllability Problems

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**Abstract:** The magnetohydrodynamic (MHD) equations in the bounded domain $\Omega \subset \mathbb{R}^d$ with the boundary $\Gamma$ are considered.

$$\partial u/\partial t - \nu\Delta u + (u\nabla)u = -\nabla p + S \cdot \operatorname{rot} B \times B,\ x \in \Omega,\ t > 0, \tag{1}$$

$$\partial B/\partial t + \operatorname{rot} E = 0,\ \operatorname{rot} B = 1/\nu_m(E + u \times B + E_c),\ \ \operatorname{div} u = 0,\ \ \operatorname{div} B = 0. \tag{2}$$

Here $u$, $B$, $E$ and $E_c$ are vector fields of velocity, magnetic induction, electric intensity and external electromotive intensity respectively; $p$ is the flow pressure, $\nu = 1/Re.$ $\nu_m = 1/R_m$, $S = M^2/Re\, R_m$, where $Re$,$Re_m$ and $M$ are the Reynolds number, Reynolds magnetic number and Hartmann number.

To the equations (1)-(2) we add the initial and the boundary value conditions

$$u|_{t=0} = u^0(x),\ B|_{t=0} = B^0(x),\ x \in \Omega,\ \ u|_\Gamma = 0,\ B \cdot n|_\Gamma = 0,\ n \times E|_\Gamma = 0, \tag{3}$$

where $n$ is the unit outward normal to the boundary.

The external electromotive intensity is considered as feedback control in the form

$$-E_c \in \partial\Phi(B). \tag{4}$$

Here $\Phi : V \to (-\infty, +\infty]$ is a convex lower semicontinuous function defined on the appropriate space of divergence-free vector functions and $\partial\Psi$ is its subdifferential.

To study the Problem (1)–(4) the theory of solvability of an evolution inequality in a Hilbert space for the operators with the quadratic nonlinearity is created. Obtained results is used for the study of MHD flows. For the 3 - dimensional flows the global in time existence of the weak solutions to the variational inequalities is proved. For the two-dimensional flows existence and uniqueness of the strong solutions are proved. Then the control problem of generation in the given moment of time the prescribed magnetic field is considered. On the base of estimates of the solution for subdifferential problem for the MHD system controllability of the flow is proven on the condition that the norm of the control is minimal.

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