The Technique of Hierarchical Matrices

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Abstract: The discretisation of partial differential equations (PDEs) leads to large systems of equations. In particular, the boundary element method (BEM) produces fully populated matrices. Several methods try to reduce the costs for storage and matrix-vector multiplication from $\mathcal{O}(n^2)$ to $\mathcal{O}(n \log^q n)$. The technique of hierarchical matrices presented here supports *all* matrix operations, i.e., addition, multiplication and inversion of matrices. This is also of interest for the sparse matrices from FEM, since the dense inverse can be computed.

Concerning the applications, we mention six topics.

BEM matrices: We generate a data-sparse approximation to the dense system matrix in order to reduce the storage requirements. The error should be comparable with the already existing discretisation error.

FEM preconditioning: Sparse FEM systems are usually solved iteratively, provided a good preconditioner is available. Instead, the technique of hierarchical matrices allows to approximate the LU-factors which are a perfect black-box preconditioner

Domain decomposition: Eliminating the interior unknowns in a domain decomposition method with non-overlapping subdomains, one obtains a dense matrix (Schur complement) for the nodal points on the skeleton. The hierarchical matrix technique allows the elimination as well as the treatment of the skeleton matrix.

Matrix equations: The Lyapunov and Riccati equation arise in control theory and define a system of n^2 equations for the n^2 unknown entries of X. Therefore the best possible solve seems to need a work of $\mathcal{O}(n^2)$. If the coefficient matrix A arises from an elliptic operator (as in control problems with a state governed by an elliptic boundary value problem), it turns out that the solution X can be well approximated by a hierarchical matrix. The costs add up to $\mathcal{O}(nk^2 \log^3 n)$ even in the case of the nonlinear Riccati equation.

Matrix functions: The matrix exponential function $\exp(-tA)$ is of general interest. We are able to compute $\exp(-tA)$ with accuracy ε with a cost of order $\mathcal{O}(n \log^p \frac{1}{\varepsilon} \log^q n)$. Similarly, other matrix functions can be computed, in particular, the sign-function $\operatorname{sign}(A)$ is a very interesting function.

Problems in high spatial dimensions: Related techniques can be applied to problems in high spatial dimensions when Kronecker products of matrices can be used. Examples are mentioned where matrices of size $N = 1024^{2048} \approx 1.2 \times 10^{6165}$ are treated.

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